Wave Functions of Bosonic Symmetry Protected Topological Phases

Cenke Xu[1] and T. Senthil[2]

[1]*Department of Physics, University of California, Santa Barbara, CA 93106*
[2]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

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We study the structure of the ground state wave functions of bosonic Symmetry Protected Topological (SPT) insulators in 3 space dimensions. We demonstrate that the differences with conventional insulators are captured simply in a dual vortex description. As an example we show that a previously studied bosonic topological insulator with both global U(1) and time-reversal symmetry can be described by a rather simple wave function written in terms of dual "vortex ribbons". The wave function is a superposition of all the vortex ribbon configurations of the boson, and a factor (-1) is associated with each self-linking of the vortex ribbons. This wave function can be conveniently derived using an effective field theory of the SPT in the strong coupling limit, and it naturally explains all the phenomena of this SPT phase discussed previously. The ground state structure for other 3d bosonic SPT phases are also discussed similarly in terms of vortex loop gas wave functions. We show that our methods reproduce known results on the ground state structure of some 2d SPT phases.

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I. INTRODUCTION

The disordered ground states of strongly interacting quantum many-body systems can have much richer structures compared with classical disordered states. The quantum richness of a system is encoded in the entanglement of its ground state wave function, and without assuming any symmetry of the Hamiltonian, the ground state wave function of a quantum many-body state can have long range entanglement, which implies that the system has a "topological order". In the last few years, motivated by the discovery of free fermion topological insulators protected by time-reversal symmetry[1–6], it was realized that short range entangled state can still be fundamentally distinct from trivial product states, as long as the system preserves certain global symmetry G. These nontrivial quantum disordered phases with short range entanglement are called "symmetry protected topological" (SPT) phases. They are separated from the trivial product state through sharp quantum phase transitions in the bulk, either continuous or first order. In space dimension $d = 1$ the Haldane spin chain provides an old and nice example of an SPT phase[7,8]. It has a bulk gap and no fractional excitations but nevertheless has dangling symmetry protected spin-1/2 moments at the edge[9–12]. The Haldane chain thus provides an early example of an interacting topological insulator.

In this paper we are mainly concerned with three dimensional symmetry protected topological insulators of bosons/spin systems. A formal mathematical classification[13] of SPT phases based on group cohomology allows a number of such phases to exist (depending on the global symmetry) but sheds little light on the physical properties. The latter have been discussed recently in Ref. 14. A characteristic feature of all SPT phases is the presence of non-trivial surface states at the interface with a trivial insulator. Indeed though the bulk is gapped and has no fractional excitations or topological order, such an interface cannot be in a trivial insulating state. Ref. 14 described the effective surface theory for a number of three dimensional bosonic topological insulators and determined the structure of the allowed non-trivial phases. These either spontaneously break the defining global symmetry or if gapped have surface topological order. Exotic symmetry preserving gapless states were also shown to be possible. A key feature is that the surface effective field theory realizes symmetry in a manner not possible in strictly two dimensional systems. Bulk topological field theories and effective field theory descriptions have also been provided.

In this work we will study the structure of the ground state wave function of various such 3d bosonic SPT insulating phases with global $U(1)$ and time reversal (Z_2^T) symmetries. The differences with conventional Mott insulators are conveniently captured in a dual description in terms of closed vortex loops. In Mott insulating phases (conventional or topological) the vortex loops have proliferated and the ground state wavefunction can be described as a vortex loop gas (see Sec. II). We show that when compared with the conventional insulator this vortex loop gas has extra phase factors depending on the topology of the vortex loop configuration. We demonstrate that these wave functions simply capture all the major phenomena of the SPT phases, both in the bulk and at the boundary, that were discussed in Ref. 14. As a key example, in Sec. III we discuss a non-trivial SPT phase with symmetry either direct ($U(1) \times Z_2^T$) (as is appropriate for a spin model realization of an interacting boson system) or semidirect ($U(1) \rtimes Z_2^T$) product. Here the vortex lines should be viewed as ribbons with a non-zero thickness and there is a phase -1 associated with each self-linking of a vortex ribbon. We briefly also discuss a different SPT phase that occurs for $U(1) \times Z_2^T$ where each vortex loop can be viewed as a $1d$ Haldane

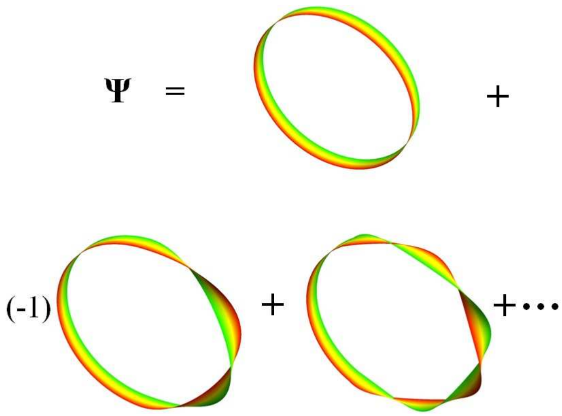


FIG. 1: The wave function of the 3d bosonic SPT discussed in this paper is a superposition of all the configurations of vortex ribbons with factor (-1) associated with each self-linking.

spin chain. These results are obtained by analysing both the sigma model effective field theory and the topological "BF" effective field theories proposed in Ref. 14 for these phases.

In $2d$ a result with a similar flavor has been derived by Levin and Gu[15] for an SPT phase with Ising, *i.e* Z_2, symmetry in terms of a domain wall loop gas with phase factors. In Sec. IV we reproduce this result using our methods. We also discuss the ground state wavefunction structure of the 2d boson topological insulator with $U(1) \times Z_2^T$ symmetry. We use this to obtain a dual vortex description of this state, and show that the physics is correctly captured.

II. WAVE FUNCTION OF TRIVIAL 3D BOSE MOTT INSULATOR

Let us start with briefly reviewing the trivial Mott insulating phase of bosons. This is conveniently modeled by a quantum disordered phase of interacting U(1) rotors on a 3d lattice, which is described by the Hamiltonian $H = \sum_{<i,j>} -t\cos(\theta_i - \theta_j) + U(\hat{n}_i)^2$. The boson creation operator $b_i = e^{i\theta_i}$ and n_i is the corresponding $U(1)$ charge at site i. θ_i and n_i are canonically conjugate. The quantum disordered phase of the rotors is equivalent to the familiar Mott insulator phase and occurs when $t/U \ll 1$. In the strong coupling limit $t \to 0$, the ground state wave function is a trivial direct product state:

$$|\Psi\rangle = \prod_i |\hat{n}_i = 0\rangle \sim \prod_i \int_0^{2\pi} d\theta_i |\theta_i\rangle. \quad (1)$$

The wave function of the quantum disordered phase with finite but small t/U can be derived through perturbation on wave function Eq. 1. For our purposes it is useful to consider a simple approximate form of the wave function that captures the physics of the Mott phase :

$$|\Psi\rangle \sim \int_0^{2\pi} \prod d\theta \exp[\sum_{<i,j>} -K\cos(\theta_i - \theta_j)] \prod_l |\theta_l\rangle, \quad (2)$$

where $K \sim t/U \ll 1$. This wave function is a superposition of configurations of θ_i with a weight that is the same as the Boltzman weight of the 3d classical rotor model. The standard duality formalism of the 3d classical rotor model leads to the dual representation of this wave function:

$$|\Psi\rangle \sim \int D\vec{A} \sum_{\vec{J}} \exp[-\int d^3x \frac{1}{2K}(\vec{\nabla}\times\vec{A})^2 + i2\pi\vec{A}\cdot\vec{J}] \times |\vec{A}(x), \vec{J}(x)\rangle, \quad (3)$$

Vector field $\vec{J}$ takes only integer values on the dual lattice, and it represents the vortex loop in the phase θ. In order to guarantee the gauge invariance of $\vec{A}$, $\vec{J}$ must have no source in the bulk: $\vec{\nabla}\cdot\vec{J} = 0$. The vortex loop $\vec{J}$ can only end at the boundary, which corresponds to a 2d vortex. The U(1) gauge field $\vec{A}$ induces long range interactions between vortex loops with coupling strength K. In the limit $K \to 0$, *i.e.* the strong coupling limit of the original rotor, the wave function Eq. 3 for quantum disordered lattice bosons becomes a equal weight superposition of all vortex loop configurations, with a weak long range interaction.

Quite generally the Mott insulating phase is obtained when the vortex loops have proliferated. Consequently the ground state wave function can be described as a loop gas of oriented interacting vortex loops. The discussion above provides a derivation of this loop gas wave function starting from a simple but approximate microsopic boson wave function. A crucial point about the structure of the loop gas wave function for the trivial Mott insulator is that it has *positive* weight for all loop configurations.

III. WAVE FUNCTION OF 3D BOSONIC SPT PHASES

A 3d SPT phase with U(1) symmetry is also a quantum disordered phase of rotor θ_i, thus it is expected that its wave function is still a superposition of vortex loop configurations. However, more physics needs to be added to the vortex loops in order to capture the novel physics of the SPT phase. One of the central results of this paper is to determine the structure of this vortex loop gas wave function for the 3d SPT phases with U(1) and time-reversal symmetry discussed in Ref. 14. We first focus on one example which occurs for both $U(1) \times Z_2^T$ and for $U(1) \rtimes Z_2^T$. We show that the ground state is described by a superposition of vortex loop configurations $|C_v\rangle$, but each vortex loop should be viewed as a "ribbon" rather than a line, and a self-linking of this ribbon contributes

exactly factor (-1) (Fig. 1):

$$|\Psi\rangle \sim \sum_{C_v} (-1)^{N_t} \psi_0 \left[C_v\right] |C_v\rangle, \quad (4)$$

where N_t is the number of self-linkings. Here $\psi_0 \left[C_v\right]$ is the weight of that vortex configuration in a trivial Mott insulator. The self-linking of a vortex ribbon is the linking number between the loops defined by the 2 ends of the ribbon.

This wave function Eq. 4 explains the key phenomena of the 3d SPT phase discussed in Ref. 14. In Ref. 14, using a 2+1d boundary field theory, it was proved that the vortex of the U(1) rotor at the boundary of this SPT is a fermion. A vortex at the boundary is the end (source) of the vortex ribbon in the bulk, and as was discussed in Ref. 16, exchanging the ends of ribbons is equivalent to twisting one of the ribbons by 2π, which according to Eq. 4 should contribute factor (-1). Thus the bulk wave function Eq. 4 already implies that the vortex at the boundary must be a fermion.

The wave function Eq. 4 can be derived either using a bulk non-linear sigma model effective field theory for the SPT phase or using a bulk topological "BF" field theory. We will present both these derivations below. In order to describe the 3d SPT with either $U(1) \rtimes Z_2^T$ or $U(1) \times Z_2^T$ symmetry, Ref. 14 proposed the following nonlinear Sigma model that involves a five component unit vector $\vec{n} = (n^1, \cdots, n^5)$, with a topological $\Theta-$term at $\Theta = 2\pi$:

$$S = \int d^3x d\tau \ \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{\Omega_4} \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_\tau n^e, \quad (5)$$

where Ω_4 is the volume of a four dimensional sphere with unit radius. Eq. 5 has an enlarged SO(5) symmetry, but later we will reduce this symmetry down to physical $U(1) \rtimes Z_2^T$ or $U(1) \times Z_2^T$.

In 3+1d, an order-disorder phase transition occurs while tuning g. We will focus on the quantum disordered phase with strong coupling g. Since $\Theta = 2\pi$ in Eq. 5, its quantum disordered phase has the same bulk spectrum as the case with $\Theta = 0$. Thus coupling constant g flows to infinity in the quantum disordered phase and this is the limit we will focus on in this paper.

The physical meaning of the $\Theta-$term in a NLSM is usually interpreted as a factor $\exp(i\Theta)$ attached to every instanton event in the space-time. Then this interpretation would lead to the conclusion that $\Theta = 2\pi$ is equivalent to $\Theta = 0$. However, this interpretation is very much incomplete, because it only tells us that theories with $\Theta = 2\pi$ and 0 have the same partition function when the system is defined on a compact manifold. These two theories actually have very different ground state wave functions. In order to expose the wave function, we need to keep an open boundary of time. In this case the wave function can be derived using the following path integral:

$$\langle \vec{n}(x)|\Psi\rangle\langle\Psi|\vec{n}'(x)\rangle \sim \int D\vec{n}(x,\tau) \exp(-S)_{\vec{n}_{\tau=+\infty}=\vec{n}', \vec{n}_{\tau=-\infty}=\vec{n}}. \quad (6)$$

The ground state wave function $|\Psi\rangle$ can then be obtained straightforwardly in the strong coupling limit $g \to +\infty$:

$$\begin{aligned} |\Psi\rangle &\sim \int D\vec{n}(x) W[\vec{n}] |\vec{n}(x)\rangle \\ W[\vec{n}] &= e^{\frac{i2\pi}{\Omega_4}\int d^3x \int_0^1 du \ \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_u n^e}. \end{aligned} \quad (7)$$

Here $\vec{n}(x,u)$ is an extension of the real space configuration $\vec{n}(x)$ that satisfies $\vec{n}(x,0) = (0,0,0,0,1)$, and $\vec{n}(x,1) = \vec{n}(x)$. Eq. 7 is a superposition of all the configurations of the O(5) vector field $\vec{n}(x)$, with a weight that is proportional to the real space Wess-Zumino-Witten (WZW) term $W[\vec{n}]$ at level-1. Thus the ground state wave function of Eq. 5 with $\Theta = 2\pi$ is fundamentally different from the case with $\Theta = 0$.

Now let us reduce the artificial SO(5) symmetry of Eq. 5 to $U(1)$ and Z_2^T. We decompose the five component vector $\vec{n}$ as $\vec{n} = (\sin(\alpha)\vec{\phi}, \ \cos(\alpha)\phi_0)$, where $\vec{\phi}$ is a unit four component vector, and $\phi_0 = \pm 1$ is an Ising order parameter. We further define two bosonic rotor operators $b_1 \sim \phi_1 + i\phi_2$, $b_2 \sim \phi_3 + i\phi_4$. Under the U(1) and Z_2^T, we take these variables to transform as

$$\begin{aligned} Z_2^T \ &: \ b_1, b_2 \to b_1, b_2 \ (U(1) \rtimes Z_2^T), \\ & \quad\ b_1, b_2 \to -b_1^*, -b_2^* \ (U(1) \times Z_2^T), \\ & \quad\ \phi_0 \to -\phi_0, \\ U(1) \ &: \ b_1 \to e^{i\theta} b_1, \quad b_2 \to e^{i\theta} b_2. \end{aligned} \quad (8)$$

We assume the system favors $\vec{\phi}$ over ϕ_0. If the time-reversal symmetry is preserved, namely $\langle\phi_0\rangle = 0$, the WZW term in the wave function Eq. 7 reduces to a theta term for the 4-component unit vector $\vec{\phi}$ in $3+0$ dimensions. Thus we get the following wave function:

$$\begin{aligned} |\Psi\rangle &\sim \int D\vec{\phi}(x) \\ &\times \ \exp(\int d^3x \frac{i\pi}{12\pi^2} \epsilon_{abcd} \epsilon_{\mu\nu\rho} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\rho \phi^d) |\vec{\phi}(x)\rangle \\ &= \int D\vec{\phi}(x) \ (-1)^{N_s} |\vec{\phi}(x)\rangle. \end{aligned} \quad (9)$$

This wave function is a superposition of all configurations of $\vec{\phi}(x)$ in real space, with a $\theta-$term defined in 3d real space, at precisely $\theta = \pi$. N_s is the Skyrmion number of the four component vector $\vec{\phi}$, since $\pi_3[S^3] = \mathbb{Z}$. The fact $\theta = \pi$ in the wave function is protected by time-reversal symmetry Z_2^T. If this Z_2^T symmetry is broken, θ in this wave function will be tuned away from π.

For our purposes we need to introduce anisotropies that reduce the symmetry from $O(4)$ to $U(1) \times U(1)$. Then the θ term (at $\theta = \pi$) implies that there is a phase

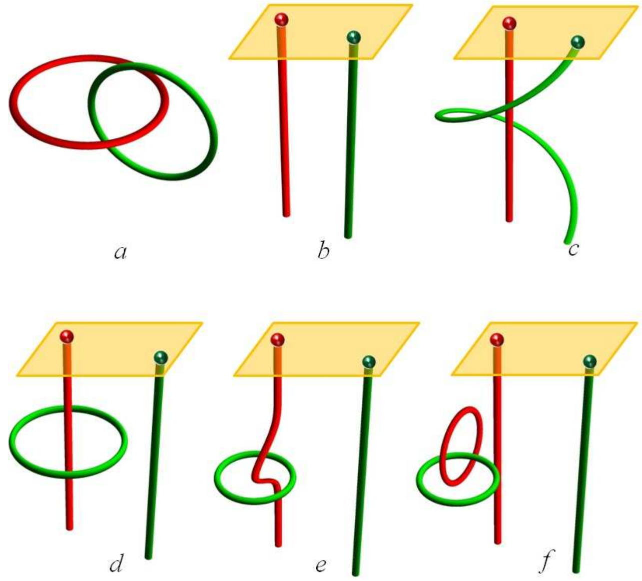


FIG. 2: (a). When the symmetry is $U(1)\times U(1)$ the bulk wave function is a superposition of two flavors of vortex loops with factor (-1) attached to each linking. $(b-f)$, braiding between two flavors of vortices at the boundary effectively creates one extra linking to the bulk vortex loops, which according to the bulk wave function would contributes factor (-1). This implies that the two flavors of vortices at the boundary have mutual semion statistics.

factor -1 each time the vortex loops of the two boson species link[17]. Thus this two species boson Mott insulator has a wave function which is a superposition of all vortex loops of the two species with a crucial factor of $(-1)^L$ where L is the total number of linked opposite species vortex loops. In contrast for the trivial Mott insulator of the two boson species system, the weight for all vortex loop configurations can be taken to be positive.

It is implicit in the discussion in terms of a four component unit vector $\vec{\phi}$ that classical configurations of the $b_{1,2}$ fields are always such that $b_{1,2}$ cannot simultaneously vanish. As the amplitude of either of these fields vanishes in their vortex core this implies that the vortex loops of the two species cannot intersect. Thus a configuration with a linking of the two vortex loops cannot be deformed to one without a linking.

This bulk wave function Eq. 9 also implies that at the 2d boundary, the vortex of b_1 and vortex of b_2 (sources of vortex loops) have a mutual semion statistics, because when one flavor of vortex encircles another flavor through a full circle, the bulk vortex loops effectively acquire one extra linking (Fig. 2), which according to the bulk wave function would contribute factor (-1).

Let us now provide an alternate derivation of this result using the bulk topological BF theory for the SPT phase also proposed in Ref. 14. This theory takes the form

$$2\pi\mathcal{L}_{3D} = \sum_I \epsilon^{\mu\nu\lambda\sigma} B^I_{\mu\nu}\partial_\lambda a^I_\sigma + \Theta \sum_{I,J} \frac{K_{IJ}}{4\pi}\epsilon^{\mu\nu\lambda\sigma}\partial_\mu a^I_\nu \partial_\lambda a^J_\sigma \tag{10}$$

Here $B^I_{\mu\nu}$ is a rank-2 antisymmetric tensor that is related to the current of boson of species $I = 1, 2$ through $j^I_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda\sigma}\partial_\nu B^I_{\lambda\sigma}$. a^I_μ is a 1-form gauge field which describes the vortices of the bosons. Specifically the magnetic field lines of a^I are identified with the vortex lines of the boson of species I. For the SPT state of interest the K matrix is simply σ_x. The parameter $\Theta = \pi$ (not to be confused with the theta parameter in the sigma model description). The crucial difference with the trivial Mott insulator is the second Θ term. To get the ground state wavefunction we again evaluate the Euclidean path integral with open temporal boundary conditions. Using the well known fact that the Θ term is the derivative of a Chern-Simons term we end up with the following ground state wave functional:

$$\psi\left[a^I_i, B^{IJ}_{jk}\right] \sim e^{i\frac{\Theta}{8\pi^2}\int d^3x \epsilon_{ijk} K^{IJ} a^I_i \partial_j a^J_k}\psi_0\left[a^I_i, B^{IJ}_{jk}\right] \tag{11}$$

Here ψ_0 is the wave functional for the trivial Mott insulator. The wave functional for the SPT insulator is thus modified by a phase factor given by a $3+0$ dimensional Chern-Simons term. As is well known the Chern-Simons term is related to a counting of the total linking number of the configuration of the magnetic flux lines of the gauge fields. Specializing to the case at hand we see that in the presence of a 2π flux line of a_1, there is a phase factor $\Theta = \pi$ whenever a 2π flux line of a_2 links with it. Thus we reproduce the result that there is a phase of π associated with each linking of opposite species vortex lines.

Finally if the $U(1)\times U(1)$ symmetry is broken down to diagonal U(1), then the vortex loops of the two species will be confined to each other. The resulting common vortex loop of the rotor $b \sim b_1 \sim b_2$ becomes a ribbon, whose two edges are the vortex loops of b_1 and b_2. Further for simplicity we assume that there is an energetic constraint at short distances that prevent two vortex lines of the same species from approaching each other. In particular we assume that the binding length scale of the opposite species vortex loops is smaller than the allowed separation between same species vortex loops. Then the vortex ribbons cannot intersect each other. Such a "hard-core" constraint on the short distance physics should not affect the universal long distance behavior of the wavefunction[24]. Note that the binding of the two species of vortex loops gives a physical implementation of the mathematical concept of 'framing' used to describe the topology of knots. The linking between the two flavors of vortex loops becomes a self-linking of the ribbon. Thus wave function Eq. 9 reduces to wave function Eq. 4. As we mentioned before, this bulk wave function Eq. 4 implies that the end point of a vortex ribbon at a 2d boundary is a fermion. Similarly bulk external sources for vortex ribbons will also be fermions.

All these results concur with the boundary theory discussed in Ref. 14. There a boundary field theory for the SPT is derived, which is a 2+1d NLSM with four component vector $\vec{\phi}$, and there is a 2+1d space-time $\Theta-$term at precisely $\Theta = \pi$. This space-time $\Theta-$term implies that the vortex of b_2 carries $1/2-$charge of b_1, and vice versa. Thus vortices of b_1 and b_2 have a mutual semion statistics. When the symmetry is broken down to one single U(1), the 2d vortex at the boundary becomes a bound state of the two flavors of vortices: thus eventually this bound vortex becomes a fermion.

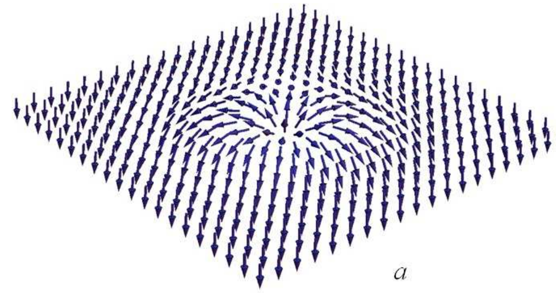
Ref. 14 also described a different interesting SPT phase with $U(1) \times Z_2^T$ symmetry. There the surface theory is such that the surface vortex carries a Kramers doublet in its core. A bulk effective field theory of this phase is also obtained[14] by starting with the $O(5)$ non-linear sigma model (Eq. 5) with anisotropies but with a different realization of symmetry from the one described above. For instance, we can decompose five component vector $\vec{n}$ in a different way: $\vec{n} = (\mathrm{Re}[b], \mathrm{Im}[b], N^x, N^y, N^z)$, where b is a rotor field that transforms under Z_2^T: $b \to -b^*$. $\vec{N}$ is a three component vector that changes sign under Z_2^T but is uncharged under the global $U(1)$. The 3+1d $\Theta-$term in Eq. 5 implies that the vortex loop of b is in a 1d Haldane phase of vector $\vec{N}$. We may now break the symmetry down to just $U(1) \times Z_2^T$. Then a vortex at the boundary must carry a Kramer's doublet, because it is effectively the edge of the 1d Haldane-like phase (with Z_2^T symmetry) along the vortex loop. This is the "Phase 1 SPT" with symmetry $U(1) \times Z_2^T$ discussed in Ref.[14].

Let us now consider the ground state wave function for this phase which in terms of the 5-component vector is still given by Eqn. 7. Now the interpretation of the WZW term is different. As is familiar from discussions of deconfined quantum criticality in terms of sigma models with WZW terms[17–19] in $2+1$ dimensions, with these symmetries an external source for a vortex line carries spin-1/2 of the $O(3)$ rotation that acts on the $\vec{N}$ vector. If the $O(3)$ symmetry is broken down to Z_2^T, then the vortex source still has a Kramers doublet. This of course is completely consistent with the picture that each vortex may be viewed as a Haldane chain. Thus the ground state wave function in this case can be viewed as a vortex loop gas of Haldane chains.

IV. GROUND STATE WAVE FUNCTION OF 2D SPT PHASES

A. 2d SPT phase with Z_2 symmetry

Let us now switch gears to SPT phases in 2d. We begin by making contact the work of Levin and Gu[15] on the Ising SPT phase. The simplest SPT phase in 2d has a Z_2 global symmetry. In Ref. 15, a lattice model for this phase has been discussed. The ground state wave function was argued to be a sum over all domain wall



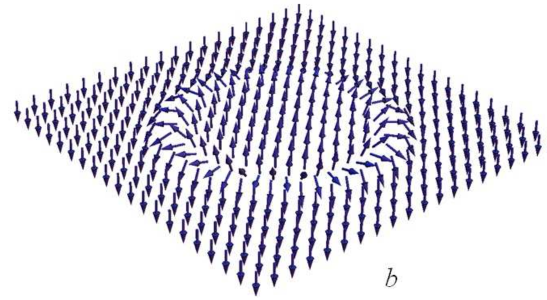


FIG. 3: (a) Skyrmion of O(3) vector $\vec{n}$ in 2d space. (b) If the SO(3) symmetry is broken down to Z_2, the Skyrmion becomes a domain wall of Z_2 order parameter n^z.

configurations with a factor $(-1)^{N_d}$ where N_d is the total number of domain wall loops. We now show how to reproduce this results within the methods of this paper.

In $2+1$ space-time dimensions many SPT phases are conveniently described by starting with an effective nonlinear sigma model field theory in terms of a four component unit vector $\vec{\phi}$. The field theory action reads

$$S = \int d^2x d\tau \ \frac{1}{g}(\partial_\mu \vec{\phi})^2 + \frac{i2\pi}{12\pi^2}\epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\rho\phi^d \quad (12)$$

The crucial ingredient is the Θ term for the 4-component unit vector at $\Theta = 2\pi$. This action Eq. 12 has an SO(4) symmetry, and this SO(4) symmetry contains a subgroup Z_2 symmetry $\vec{\phi} \to -\vec{\phi}$. Eventually we will break the artificial SO(4) symmetry of Eq. 12 down to this Z_2 subgroup.

In the paramagnetic phase, $i.e.$ in the limit $g \to +\infty$, the bulk ground state wave function is

$$|\Psi\rangle \sim \int D\vec{\phi}(x) \ \exp\{\frac{i2\pi}{12\pi^2} \times \int d^2x \int_0^1 du \ \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\rho\phi^d\} \ |\vec{\phi}(x)\rangle \quad (13)$$

which is a superposition of all configurations of $\vec{\phi}(x)$ with a real space WZW weight.

Now let us decompose the four component vector into $\vec{\phi} = (\cos(\alpha)\phi_0, \ \sin(\alpha)\vec{n})$, where $\phi_0 = \pm 1$ is an Ising order parameter, and $\vec{n}$ is a unit three component vector: $(\vec{n})^2 = 1$. We also break the SO(4) symmetry down to $Z_2 \times \mathrm{SO}(3)$ symmetry:

$$Z_2 \ : \ \phi_0 \to -\phi_0, \quad \vec{n} \to -\vec{n},$$

$$\mathrm{SO}(3)\ :\ \text{Rotation of } \vec{n}. \tag{14}$$

Under this symmetry reduction, if the system energetically favors vector $\vec{n}$ over ϕ_0 (favors $\alpha = \pi/2$), the wave function Eq. 13 reduces to

$$|\Psi\rangle \ \sim \ \int D\vec{n}(x)\ \exp\{\frac{i\pi}{8\pi}\int d^2x\ \epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c\}\ |\vec{n}(x)\rangle$$
$$= \ \sum_{N_s}(-1)^{N_s}|\vec{n}(x)\rangle, \tag{15}$$

where N_s is the number of Skyrmions of O(3) vector $\vec{n}$ in the 2d space. As long as we keep the Z_2 symmetry $\vec{\phi} \rightarrow -\vec{\phi}$, the expectation value of ϕ_0 is zero, then each Skyrmion will contribute a phase factor exactly (-1).

Now let us further break $Z_2 \times \mathrm{SO}(3)$ symmetry down to $Z_2 \times \mathrm{SO}(2)$:

$$Z_2\ :\ \phi_0 \rightarrow -\phi_0, \quad \vec{n} \rightarrow -\vec{n},$$
$$\mathrm{SO}(2)\ :\ \text{Rotation of } n^x,\ n^y. \tag{16}$$

Also we assume that the system favors n^z over n^x and n^y, then each skyrmion becomes a domain wall of Z_2 order parameter $n^z(x)$ (Fig. 3). Now the wave function Eq. 15 reduces to a superposition of configurations of Z_2 order parameter $n^z(x)$:

$$|\Psi\rangle \sim \sum_{n_z}(-1)^{N_d}|n^z(x)\rangle, \tag{17}$$

where N_d is the number of closed domain wall loops of $n^z(x)$. Eventually we can also break the residual SO(2) symmetry, and the wave function Eq. 17 is unchanged.

The wave function Eq. 17 is exactly the one derived from the lattice model of 2d SPT phase with Z_2 symmetry[15]. In the appendix we will also demonstrate that the effective field theory Eq. 12 implies that, after coupling this SPT phase to a dynamical Z_2 gauge field, the $\pi-$flux of this Z_2 gauge field has a semion statistics, which is consistent with the result in Ref. 15.

With a full SO(4) symmetry, the edge states of Eq. 12 with precisely $\Theta = 2\pi$ is the nonchiral $\mathrm{SU}(2)_1$ conformal field theory (or equivalently as an SO(4) non-linear sigma model with a level-1 WZW term). Since the original SO(4) symmetry is reduced to its Z_2 subgroup $\vec{\phi} \rightarrow -\vec{\phi}$, we have to argue that the edge state of Eq. 12 survives under this symmetry reduction. Because the Z_2 symmetry acts on all four components of $\vec{\phi}$, in the boundary WZW model, terms allowed by the Z_2 symmetry are $\sum_{i,j} g_{ij}\phi_i\phi_j$ $(i, j = 0, 1, 2, 3)$. If these terms are relevant, it leads to spontaneous Z_2 symmetry breaking and two fold degeneracy at the boundary. Thus the edge state cannot be completely trivial (gapped and nondegenerate) as long as the Z_2 symmetry is preserved.

B. 2d SPT phase with $U(1) \rtimes Z_2^T$ symmetry

Finally we consider the 2d bosonic topological insulator which occurs when the global symmetry is $U(1) \rtimes Z_2^T$. This may be described by starting again with the same 4-component non-linear sigma model but with the following implementation of the physical symmetry. We write $\phi_2 - i\phi_3 = b$ and let b have charge-1 under the global $U(1)$. Under Z_2^T, we demand $b \rightarrow b, \phi_0 \rightarrow -\phi_0, \phi_1 \rightarrow -\phi_1$. As before we again assume first an anisotropy that prefers $\vec{n}$ over ϕ_0 so that the ground state wave function is given by Eqn. 15. Now we introduce further anisotropy to reduce to the desired $U(1) \rtimes Z_2^T$. The defects of the charged field b are of course point vortices. In the core of these vortices the amplitude of b is suppressed and the $\vec{\phi}$ points entirely in the ϕ_1 direction. There are two different vortices - known as merons - depending on whether in the core $\phi_1 = \pm 1$. Each meron may be viewed as half a skyrmion and has $N_s = \mathrm{sgn}(\phi_1)1/2$. Thus the ground state wave function is then a sum over all possible configurations of the two kinds of meron vortices with phase factors $e^{\pm i\frac{\pi}{2}}$ for the two kinds of vortex. Let $n_{v\pm}$ be the vortex number of either species at site i in a lattice description. Then we require that the total vorticity $N_v = \sum_i n_{v+} + n_{v-} = 0$. Then the phase factor in the wave function $e^{i\frac{\pi}{2}\sum_i(n_{v+}-n_{v-})} = (-1)^{\sum_i n_{v-}}$. Thus there is a relative phase of -1 associated with $-$ vortices compared to $+$ vortices.

We now argue that this structure of the wave function matches what is known about the 2d bosonic topological insulator. Consider a dual description of such an insulator. From our arguments above there are two kinds of vortex fields $\Phi_{v\pm}$ corresponding to the two meron vortices. The dual vortex theory will have a Lagrangian

$$\mathcal{L}_d = \sum_{s=\pm}|(\partial_\mu - ia_\mu)\Phi_{vs}|^2 + ... + \frac{\kappa}{2}\left(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\right)^2 \tag{18}$$

Here a_μ is the usual fluctuating non-compact $U(1)$ gauge field whose flux density is the original boson number. There must in addition be terms where the meron cores tunnel into each other:

$$\lambda\left(\Phi^\dagger_{v+}\Phi_{v-} + h.c\right) \tag{19}$$

We begin by ignoring these and we will reinstate them later. Under time reversal a vortex must go to an anti vortex (as the boson phase is odd) and the meron cores flip into each other. Thus under Z_2^T

$$\Phi_{v+} \rightarrow \pm\Phi^\dagger_{v-} \tag{20}$$

In the trivial insulator all vortex configurations contribute with the same sign and we must choose $\Phi_{v+} \rightarrow +\Phi^\dagger_{v-}$. But for the topological insulator there is a relative $-$ sign between the two vortex species. Thus we must choose $\Phi_{v+} \rightarrow -\Phi^\dagger_{v-}$. Condensing vortices that transform in this manner will give us the boson topological

insulator. Now let us include the meron core tunneling term. Then the two vortex species mix with each other so that we identify a single vortex $\Phi_v = \Phi_{v+} \sim \Phi^*_{v-}$. Its transformation under time reversal is $\Phi_v \to \pm\Phi^\dagger_v$ where the $+$ sign describes the trivial insulator and the $-$ the topological insulator. This is exactly the same transformation law for the vortices that is dictated by the edge theory analysis of the 2d boson topological insulator[20,21]. Thus the wave function description we developed captures the physics of this state, and further gives a bulk dual vortex description.

V. SUMMARY

In summary, we have demonstrated in this work that, although most of the novel phenomena of a SPT phase occur at its boundary, its bulk ground state wave function is indeed drastically different from a trivial direct product disordered phase. This bulk wave function can be conveniently derived from the effective field theory of the SPT phase. The structure of the ground state wave functions in terms of dual vortex configurations derived in this work provide a simple physical picture of the phenomena associated with these SPT phases.

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[1] C. L. Kane and E. J. Mele, Physical Review Letter **95**, 226801 (2005).
[2] C. L. Kane and E. J. Mele, Physical Review Letter **95**, 146802 (2005).
[3] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).
[4] J. E. Moore and L. Balents, Physical Review B **75**, 121306(R) (2007).
[5] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).
[6] R. Roy, Physical Review B **79**, 195322 (2009).
[7] F. D. M. Haldane, Phys. Lett. A **93**, 464 (1983).
[8] F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).
[9] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. lett. **59**, 799 (1987).
[10] T. Kennedy, J. Phys. Condens. Matter **2**, 5737 (1990).
[11] M. Hagiwara, K. Katsumata, I. Affleck, B. I. Halperin, and J. P. Renard, Phys. Rev. Lett. **65**, 3181 (1990).
[12] T.-K. Ng, Phys. Rev. B **50**, 555 (1994).
[13] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, arXiv:1106.4772 (2011).
[14] A. Vishwanath and T. Senthil (arXiv:1209.3058).
[15] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012).
[16] J. C. Y. Teo and C. L. Kane, Phys. Rev. Lett. **104**, 046401 (2010).
[17] T. Senthil and M. P. A. Fisher, Phys. Rev. B **74**, 064405 (2006).
[18] A. Tanaka and X. Hu, Phys. Rev. Lett. **95**, 036402 (2005).
[19] T. Grover and T. Senthil, Phys. Rev. Lett. **100**, 156804 (2008).
[20] M. Levin and A. Stern, Phys. Rev. B **86**, 115131 (2012).
[21] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).
[22] Z.-X. Liu and X.-G. Wen, arXiv:1205.7024 (2012).
[23] T. Senthil and M. Levin, Phys. Rev. Lett. **110**, 046801 (2013).
[24] Indeed if we do allow intersection of the ribbons then the phase factors are obtained by going back to the original $U(1) \times U(1)$ theory with the two species of vortex loops. The embedding of the $U(1)$ in the higher $U(1)\times U(1)$ symmetry gives a short distance 'regularization' of ribbon intersections. Banning intersections enables us to discuss the essential physics without complications.

Appendix A: Appendix: Dynamical Z_2 gauge fields in the 2d Ising SPT

In this appendix we demonstrate that the effective field theory Eq. 12 not only gives us the correct ground state wave function (Eq. 17) of the 2d Ising SPT phase, after coupling the SPT phase to a Z_2 gauge field, the topological $\Theta-$term of Eq. 12 also leads to nontrivial statistics of the dynamical $\pi-$flux (vison) of the Z_2 gauge field.

First of all, Eq. 12 can be rewritten as a SU(2) principle chiral model, by introducing SU(2) matrix field $G = \phi^0\sigma^0 + i\vec{\phi}\cdot\vec{\sigma}$. G has SU(2)$-$left and SU(2)$-$right transformations: $G \to V_L^\dagger G V_R$. Let us "gauge" SU(2)$-$left and SU(2)$-$right transformations with dynamical U(1) gauge fields $a_\mu\sigma^z$ and $b_\mu\sigma^z$, *i.e.* replace $\partial_\mu G$ with $\partial_\mu G + ia_\mu\sigma^z G + ib_\mu G\sigma^z$. According to Ref. 22,23, after integrating out matrix field G, gauge fields a_μ and b_μ both acquire a Chern-Simons term:

$$S_{cs} = \int d^2x d\tau \; \frac{i2}{4\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu a_\rho - \frac{i2}{4\pi}\epsilon_{\mu\nu\rho}b_\mu\partial_\nu b_\rho. \quad \text{(A1)}$$

This is because Eq. 12 also describes a U(1) bosonic SPT with Hall conductivity 2.

A dynamical U(1) gauge field with level$-k$ has the following properties: its charged quasiparticle carries gauge flux $2\pi/k$, and this quasiparticle has a statistics angle π/k. Thus the Chern-Simons action Eq. A1 gives the

$\pi-$flux of U(1) gauge field a_μ and b_μ a semion statistics, with statistics angle $+\pi/2$ and $-\pi/2$ respectively. Notice that the two U(1) gauge groups share the same Z_2 transformation $G \to -G$, thus we can break the two U(1) gauge fields down to one Z_2 gauge field, then the dynamical $\pi-$flux of this Z_2 gauge field has two different flavors with semionic statistics angle $+\pi/2$ and $-\pi/2$ respectively.

In Ref. 15, using their lattice model, the authors concluded that the dynamical $\pi-$flux of this Z_2 gauge field has a semion statistics. Here we have derived the same result using our field theory Eq. 12.